Appendix A

Derivation of the Frequentist and Bayes-modified Likelihood Functions

The notation used in this Appendix is the same as what has been introduced in the main portion of this application. In what follows, we make the following assumptions:

(i) The visibility of a particular pilot at $(x,y) \in A$ at a given time t is independent of whether the pilot was visible at any time $s<t$, and is independent of t as well.

(ii) The visibility of pilot i at time t is independent of the visibility of pilot j ($j \neq i$) at time t.

(iii) Throughout the duration of the prediction process, the mobile is relatively stationary, i.e., the mobile unit remains in its initial sub-cell.

Defining $L^0(x,y)=1$, $\forall\,(x,y)\in A$, a recursive form for the exact likelihood function through the first s measurement epochs for the unknown location of the mobile based on (i)-(iii) is

$$L^s(x,y) \propto L^{s-1}(x,y)\prod_{ij\in K}[\theta_{ij}(x,y)]^{\mu^s_{ij}}[1-\theta_{ij}(x,y)]^{1-\mu^s_{ij}} \quad , \quad (x,y)\in A. \tag{A1}$$

The functions $\theta_{ij}(x,y)$ in (A1) are unknown, so the exact likelihood function is not calculable. Replacing the $\theta_{ij}(x,y)$ with their approximations $\tilde{\theta}_{ij}(x,y)$, defined by equation (B9) in Appendix B, gives the definition of the frequentist likelihood function $L^s_{ML}(x,y)$ shown in expression (1).

The frequentist likelihood function (1) was derived under the assumption that a family of functions $\tilde{\theta}_{ij}(x,y)$ accurately approximate the unknown functions $\theta_{ij}(x,y)$. The approximations $\tilde{\theta}_{ij}(x,y)$ are based on the RF model described in Appendix B. The RF model is an approximation to a very complex stochastic process. Bayesian methods allow the uncertainty associated with $\tilde{\theta}_{ij}(x,y)$ to be incorporated into the model framework. Rather than assuming $\tilde{\theta}_{ij}(x,y)\equiv\theta_{ij}(x,y)$, we can model the parameters $\theta_{ij}(x,y)$ with

independent prior distributions that have mean values equal to $\tilde{\theta}_{ij}(x,y)$. For each $(x,y)\in A$, we use a prior distribution for $\theta_{ij}(x,y)$ that is a beta distribution with probability density function given by

$$f_{\theta_{ij}(x,y)}(u)=\frac{1}{B[\alpha_{ij}(x,y)\,,\beta_{ij}(x,y)]}u^{\alpha_{ij}(x,y)-1}(1-u)^{\beta_{ij}(x,y)-1}\quad,\quad 0\le u\le 1 \tag{A2}$$

where $B[\cdot,\cdot]$ is the complete beta function. Note that the two parameters, $\alpha_{ij}(x,y)$ and $\beta_{ij}(x,y)$, to be determined in what follows, depend on both the possible location $(x,y)\in A$ and the pilot $ij\in K$. Whereas the frequentist likelihood is defined by implicitly assuming $\Pr[\,\theta_{ij}(x,y)=\tilde{\theta}_{ij}(x,y)\,]=1$, the Bayes-modified likelihood is defined by taking the expected value of the exact likelihood (A1) with respect to the beta distributions defined by (A2). The recursive form of the exact likelihood (A1) does not amend itself to evaluating the expected value as easily as the non-recursive form of the exact likelihood which is

$$L^{s}(x,y)\propto\prod_{ij\in K}[\theta_{ij}(x,y)]^{n_{ij}^{s}}[1-\theta_{ij}(x,y)]^{s-n_{ij}^{s}}\quad,\quad (x,y)\in A \tag{A3}$$

where n_{ij}^{s} is the number of times pilot ij was visible through the first s measurement epochs. Evaluating the expected value of (A3), with respect to the beta distributions defined by (A2) gives the Bayes-modified likelihood function

$$\begin{aligned}L_{BML}^{s}(x,y)&\propto E\left\{\prod_{ij\in K}[\theta_{ij}(x,y)]^{n_{ij}^{s}}[1-\theta_{ij}(x,y)]^{s-n_{ij}^{s}}\right\}\\&=\prod_{ij\in K}\frac{B[\,n_{ij}^{s}+\alpha_{ij}(x,y)\,,s-n_{ij}^{s}+\beta_{ij}(x,y)\,]}{B[\,\alpha_{ij}(x,y)\,,\beta_{ij}(x,y)\,]}\quad,\quad (x,y)\in A.\end{aligned} \tag{A4}$$

It is easily verified that equation (A4) has the recursive form shown in the functional expression (4).

The definition of the Bayes-modified likelihood function will be complete once values for $\alpha_{ij}(x,y)$ and $\beta_{ij}(x,y)$ are specified. Determining what values to use for

$\alpha_{ij}(x,y)$ and $\beta_{ij}(x,y)$ is an example of the classic Bayesian dilemma – how to specify a prior distribution? An underlying principle in specifying a prior distribution is that any available information about the parameter of interest, in this case $\theta_{ij}(x,y)$, should be reflected in the prior distribution that is used. In this case, we have an approximation of $\theta_{ij}(x,y)$ from the RF model, namely $\tilde{\theta}_{ij}(x,y)$, which we will use as the mean of the prior distribution for $\theta_{ij}(x,y)$. Accordingly, we have the following constraint:

$$\frac{\alpha_{ij}(x,y)}{\alpha_{ij}(x,y)+\beta_{ij}(x,y)}=\tilde{\theta}_{ij}(x,y). \tag{A5}$$

Since we have two unknowns, $\alpha_{ij}(x,y)$ and $\beta_{ij}(x,y)$, we need one additional constraint to uniquely determine both $\alpha_{ij}(x,y)$ and $\beta_{ij}(x,y)$. To obtain the second constraint, we impose that the prior be as vague as possible, subject to the first constraint. To implement this second constraint, we will maximize the variance of the prior, subject to the fixed mean value. The second constraint is consistent with the quantity of the prior knowledge we have about $\theta_{ij}(x,y)$ and except for anticipating a value near $\tilde{\theta}_{ij}(x,y)$, we know nothing else about its potential value. We also restrict attention to beta distributions which have bounded density functions, implying $\alpha_{ij}(x,y)\geq 1$ and $\beta_{ij}(x,y)\geq 1$. The restriction to bounded density functions only rules out "U-shaped and "Slide-shaped" beta distributions which would not typically be intuitive representations of the prior information on $\theta_{ij}(x,y)$. The problem of selecting $\alpha_{ij}(x,y)$ and $\beta_{ij}(x,y)$ has now been sufficiently constrained so that it can be solved by using the following Lemma.

Lemma

Consider the family of beta distribution with parameters $a\geq 1$ and $b\geq 1$. The values of a and b which maximize the variance, subject to the distribution having a given mean value μ, are: $a=1$ and $b=(1-\mu)/\mu$, if $\mu\leq 1/2$; $a=\mu/(1-\mu)$ and $b=1$, if $\mu>1/2$.

Proof of Lemma

The variance of the beta distribution is given by $\sigma^2=ab/[(a+b+1)(a+b)^2]$. Since we have $\mu=a/(a+b)$, we can write $b=a(1-\mu)/\mu$, and hence $\sigma^2=\mu^2(1-\mu)/(a+\mu)$. To

maximize the variance, we need to minimize a, subject to the constraint that both $a \geq 1$ and $b \geq 1$. Clearly, if $\mu \leq 1/2$ then we can choose $a = 1$ and $b = (1-\mu)/\mu \geq 1$ will be ensured. Likewise, if $\mu > 1/2$ then the smallest we can choose a and still have $b \geq 1$ is $a = \mu/(1-\mu) \geq 1$.

Applying the Lemma with $a = \alpha_{ij}(x,y)$, $b = \beta_{ij}(x,y)$ and $\mu = \widetilde{\theta}_{ij}(x,y)$ gives the formulas in equations (2) and (3) for $\alpha_{ij}(x,y)$ and $\beta_{ij}(x,y)$.